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STATE OF STRESS OF PLATE WITH A THIN-WALLED INCLUSION ALONG THE ARC OF A CIRCLE*

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An approach, different in comparison to /1/, is proposed for the investigation of the influence of a thin-walled elastic inclusion along the arc of a circle on the state of stress of a homogeneous isotropic plate. The solution of the problem reduces to a system of two singular integro-differential equations of Prandtl type. A numerical analysis is presented for the stress intensity coefficients.

1. Let us consider the equilibrium of an isotropic plate with a thin-walled elastic inclusion of constant width along the arc of a circle of radius R. We assume that the plate is subjected to uniformly distributed stresses σ_1 and σ_2 at infinity (Fig.1).

Let $2h$ denote the width of the inclusion, 2φ the aperture angle, L is the arc of the circle with central angle 2φ, and a and b the lower and upper ends of the inclusion, respectively. We ascribe the subscript 0 to quantities characterizing the thin-walled inclusion. The boundary value of the function on the edge of the inclusion that is closer to the center of the circle will be denoted with a plus sign, while the other edge will be denoted with a minus.

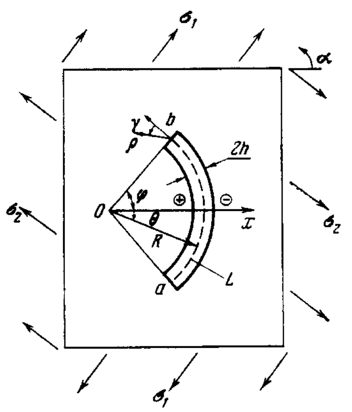

Fig.1

We assume that the following boundary conditions can hold on the edges of the inclusion

$$(\sigma_{rr}+i\tau_{r\vartheta})_0^{\pm}=(\sigma_{rr}+i\tau_{r\vartheta})^{\pm},\quad (V_r+iV_\vartheta)_0^{\pm}=(V_r+iV_\vartheta)^{\pm} \tag{1.1}$$

The components σ_{rr}, $\sigma_{\vartheta\vartheta}$, $\tau_{r\vartheta}$ of the stress tensor, and the components V_r and V_ϑ of the displacement vector under plane elasticity theory conditions are expressed in terms of two analytic functions $\Phi(z)$ and $\Omega(z)$ by means of the following formulas /2/

$$\Phi(z)+\frac{R^2}{r^2}\Omega\left(\frac{R^2}{\bar z}\right)+\left(1-\frac{R^2}{r^2}\right)[\overline{\Phi(z)}-\bar z\overline{\Phi'(z)}]=m \tag{1.2}$$

$$\varkappa\Phi(z)-\frac{R^2}{r^2}\Omega\left(\frac{R^2}{\bar z}\right)-\left(1-\frac{R^2}{r^2}\right)[\overline{\Phi(z)}-\bar z\overline{\Phi'(z)}]=2\mu n$$

$$m=\sigma_{rr}+i\tau_{r\vartheta},\quad n=\frac{e^{i\vartheta}}{iz}\left[\frac{\partial}{\partial\vartheta}(V_r+iV_\vartheta)+i(V_r+V_\vartheta)\right]$$

$$\varkappa=\frac{3-\nu}{1+\nu}$$

(ν is the Poisson's ratio). The following expansions are valid for the functions $\Phi(z)$ and $\Omega(z)$:

$$\Phi(z)=\Gamma+O\left(\frac{1}{z^2}\right),\quad |z|>1;\quad \Omega(z)=B_0-\frac{\overline{\Gamma'}R^2}{z^2}+O(z),|z|<1 \tag{1.3}$$

$$\Gamma=\frac{1}{4}(\sigma_1+\sigma_2),\quad \Gamma'=-\frac{1}{2}(\sigma_1-\sigma_2)e^{-2i\alpha},\quad B_0=\overline{\Phi(0)}$$

Using the formulas (1.2) and neglecting higher order quantities of smallness as compared with h, we can write for the thin-walled inclusion

$$m_0^+-m_0^-=2i\frac{h}{R}A(t),\quad t\in L \tag{1.4}$$

$$m_0^++m_0^-=2\left[\Phi_0(t)+\overline{\Phi_0(t)}+\frac{R^2}{t^2}\Gamma_0(t)\right],\quad t\in L$$

$$n_0^+-n_0^-=\frac{ih}{R\mu_0}B(t),\quad t\in L$$

$$n_0^++n_0^-=\frac{1}{\mu_0}\left[\varkappa_0\Phi_0(t)-\overline{\Phi_0(t)}-\frac{R^2}{t^2}\Gamma_0(t)\right]+2i\varepsilon,\quad t\in L$$

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$$A(t)=\frac{\partial}{\partial\vartheta}\Phi_0(t)+\frac{\partial}{\partial\vartheta}\overline{\Phi_0(t)}-\frac{R^2}{t^2}\frac{\partial}{\partial\vartheta}\Gamma_0(t)$$
$$B(t)=\varkappa_0\frac{\partial}{\partial\vartheta}\Phi_0(t)-\frac{\partial}{\partial\vartheta}\overline{\Phi_0(t)}+\frac{R_2}{t^2}\frac{\partial}{\partial\vartheta}\Gamma_0(t)$$

Here $\Phi_0(t)$ and $\Gamma_0(t)$ and unknown functions to be determined, and ε is the angle of turning of the inclusion as a rigid whole.

We refer the boundary conditions from the edges of the inclusion to the line L for the plate. Using the relationships (1.1), (1.2), and (1.4), we arrive at the following boundary value problems to determine the piecewise-holomorphic functions $\Phi(z)$ and $\Omega(z)$ with the line of jumps L:

$$[\Phi(t)-\Omega(t)]^+-[\Phi(t)-\Omega(t)]^-=2i\frac{h}{R}A(t),\quad t\in L \tag{1.5}$$

$$[\varkappa\Phi(t)+\Omega(t)]^+-[\varkappa\Phi(t)+\Omega(t)]^-=2i\frac{h}{R}\frac{\mu}{\mu_0}B(t),\quad t\in L$$
$$[\Phi(t)+\Omega(t)]^++[\Phi(t)+\Omega(t)]^-=2\left[\Phi_0(t)+\overline{\Phi_0(t)}+\frac{R^2}{t^2}\Gamma_0(t)\right],\quad t\in L \tag{1.6}$$
$$\varkappa[\Phi^+(t)+\Phi^-(t)]-[\Omega^+(t)+\Omega^-(t)]=2\frac{\mu}{\mu_0}\left[\varkappa_0\Phi_0(t)-\overline{\Phi_0(t)}-\frac{R^2}{t^2}\Gamma_0(t)\right]+4i\mu\varepsilon,\quad t\in L$$

Solving the linear conjugate problem (1.5), we find

$$\Phi(z)=\frac{h}{\pi R(1+\varkappa)}\left[\int_a^b\frac{A(t)}{t-z}dt+\frac{\mu}{\mu_0}\int_a^b\frac{B(t)}{t-z}dt\right]+\Gamma \tag{1.7}$$
$$\Omega(z)=\frac{h}{\pi R(1+\varkappa)}\left[-\varkappa\int_a^b\frac{A(t)}{t-z}dt+\frac{\mu}{\mu_0}\int_a^b\frac{B(t)}{t-z}dt\right]+B_0-\frac{\bar\Gamma' R^2}{z^2}$$

Substituting (1.7) into the relationships (1.6), we obtain the following system to two singular Prandtl-type integro-differential equations to determine the unknown functions $\Phi_0(t)$ and $\Gamma_0(t)$:

$$\Phi_0(t)+\overline{\Phi_0(t)}+\frac{R^2}{t^2}\Gamma_0(t)+\frac{a_1}{\pi}\int_a^b\frac{\partial}{\partial\vartheta}\Phi_0(u)\frac{du}{u-t}+\frac{a_2}{\pi}\int_a^b\frac{\partial}{\partial\vartheta}\overline{\Phi_0(u)}\frac{du}{u-t}+\frac{a_3}{\pi}\int_a^b\frac{R^2}{u^2}\frac{\partial}{\partial\vartheta}\Gamma_0(u)\frac{du}{u-t}=\Gamma+B_0-\frac{\bar\Gamma' R^2}{t^2},\quad t\in L \tag{1.8}$$
$$\frac{\mu}{\mu_0}\left[\varkappa_0\Phi_0(t)-\overline{\Phi_0(t)}-\frac{R^2}{t^2}\Gamma_0(t)\right]+\frac{b_1}{\pi}\int_a^b\frac{\partial}{\partial\vartheta}\Phi_0(u)\frac{du}{u-t}+\frac{b_2}{\pi}\int_a^b\frac{\partial}{\partial\vartheta}\overline{\Phi_0(u)}\frac{du}{u-t}+\frac{b_3}{\pi}\int_a^b\frac{R^2}{u^2}\frac{\partial}{\partial\vartheta}\Gamma_0(u)\frac{du}{u-t}=\varkappa\Gamma-B_0+\frac{\bar\Gamma' R^2}{t^2}-2i\mu\varepsilon,\quad t\in L$$

$$a_1=-\frac{h}{R(1+\varkappa)}\left(2\frac{\mu}{\mu_0}\varkappa_0+1-\varkappa\right),$$
$$a_2=-a_3=\frac{h}{R(1+\varkappa)}\left(2\frac{\mu}{\mu_0}-1+\varkappa\right)$$
$$b_1=-\frac{h}{R(1+\varkappa)}\left[2\varkappa+\frac{\mu}{\mu_0}(\varkappa-1)\varkappa_0\right],$$
$$b_2=-b_3=\frac{h}{R(1+\varkappa)}\left[2\varkappa-\frac{\mu}{\mu_0}(\varkappa-1)\right]$$

On the basis of (1.7) and (1.3) we arrive at the following conditions:

$$\int_a^b\frac{\partial}{\partial\vartheta}\Phi_0(t)\,dt=0,\quad \int_a^b\left[\frac{\partial}{\partial\vartheta}\overline{\Phi_0(t)}-\frac{R^2}{t^2}\frac{\partial}{\partial\vartheta}\Gamma_0(t)\right]dt=0 \tag{1.9}$$

that express the vanishing of the coefficient of $1/z$ in the expansions (1.3).

Executing the substitution

$$t = \frac{x+i\beta}{-x+i\beta}R, \quad \beta = -\operatorname{ctg}\varphi/2, \quad |x| \leqslant 1$$

in (1.8) and (1.9), we obtain

$$\frac{a_1}{\pi}\int_{-1}^{1}\frac{\Phi_0'(v)}{v-x}dv + \frac{a_2}{\pi}\int_{-1}^{1}\frac{\overline{\Phi_0'(v)}}{v-x}dv + \frac{a_3}{\pi}\left(\frac{-x+i\beta}{x+i\beta}\right)^2\int_{-1}^{1}\frac{\Gamma_0'(v)}{v-x}dv - \tag{1.10}$$
$$\frac{2\beta}{x^2+\beta^2}\left[\Phi_0(x) + \overline{\Phi_0(x)} + \left(\frac{-x+i\beta}{x+i\beta}\right)^2\Gamma_0(x)\right] +$$
$$\frac{1}{\pi}\frac{x-i\beta}{x+i\beta}\{a_1[\Phi_0(1)-\Phi_0(-1)] + a_2[\overline{\Phi_0(1)} - \overline{\Phi_0(-1)}]\} +$$
$$\frac{a_3}{\pi}\frac{(x-i\beta)^3}{(x^2+\beta^2)^2}[\Gamma_0(1)-\Gamma_0(-1)] +$$
$$\frac{2ia_3\beta}{\pi}\left(\frac{x-i\beta}{x^2+\beta^2}\right)^2\int_{-1}^{1}\frac{t-i\beta}{t+i\beta}\Gamma_0'(t)\,dt =$$
$$-\frac{2\beta}{x^2+\beta^2}\left[\Gamma + B_0 - \overline{\Gamma}'\left(\frac{-x+i\beta}{x+i\beta}\right)^2\right], \quad |x| \leqslant 1$$
$$\frac{b_1}{\pi}\int_{-1}^{1}\frac{\Phi_0'(v)}{v-x}dv + \frac{b_2}{\pi}\int_{-1}^{1}\frac{\overline{\Phi_0'(v)}}{v-x}dv + \frac{b_3}{\pi}\left(\frac{-x+i\beta}{x+i\beta}\right)^2\int_{-1}^{1}\frac{\Gamma_0'(v)}{v-x}dv -$$
$$\frac{2\beta}{x^2+\beta^2}\frac{\mu}{\mu_0}\left[\varkappa_0\Phi_0(x) - \overline{\Phi_0(x)} - \left(\frac{-x+i\beta}{x+i\beta}\right)^2\Gamma_0(x)\right] +$$
$$\frac{1}{\pi}\frac{x-i\beta}{x+i\beta}\{b_1[\Phi_0(1)-\Phi_0(-1)] + b_2[\overline{\Phi_0(1)} - \overline{\Phi_0(-1)}]\} +$$
$$\frac{b_3}{\pi}\frac{(x-i\beta)^3}{(x^2+\beta^2)^2}[\Gamma_0(1)-\Gamma_0(-1)] + \frac{2ib_3\beta}{\pi}\left(\frac{x-i\beta}{x^2+\beta^2}\right)^2 \times$$
$$\int_{-1}^{1}\frac{t-i\beta}{t+i\beta}\Gamma_0'(t)\,dt =$$
$$-\frac{2\beta}{x^2+\beta^2}\left[\varkappa\Gamma - B_0 - \overline{\Gamma}'\left(\frac{-x+i\beta}{x+i\beta}\right)^2 - 2i\mu\varepsilon\right], \quad |x| \leqslant 1$$

$$\int_{-1}^{1}\frac{x+i\beta}{x-i\beta}\Phi_0'(x)\,dx = 0, \quad \int_{-1}^{1}\left[\frac{x+i\beta}{x-i\beta}\overline{\Phi_0'(x)} - \frac{x-i\beta}{x+i\beta}\Gamma_0'(x)\right]dx = 0 \tag{1.11}$$

2. Following /3/, we seek the solution of the system of equations (1.10) and (1.11) in the form

$$\Phi_0(x) = K_0 + x_0 \arcsin x - \sqrt{1-x^2}\sum_{m=1}^{\infty}\frac{x_m U_{m-1}(x)}{m} \tag{2.1}$$
$$\Gamma_0(x) = M_0 + y_0 \arcsin x - \sqrt{1-x^2}\sum_{m=1}^{\infty}\frac{y_m U_{m-1}(x)}{m}$$

where $U_m(x)$ are Chebyshev polynomials of the second kind, and $x_m, y_m\ (m = 0, 1, \ldots), K_0, M_0$ are unknown complex coefficients.

Substituting (2.1) into (1.7), we find

$$\Phi(z) = -\frac{2ih\cos(\varphi/2)\,z}{(z+R)(1+\varkappa)X_0(z)}\sum_{m=0}^{\infty}\left[N_m(z) + \frac{\mu}{\mu_0}R_m(z)\right] \times \tag{2.2}$$
$$T_m\left(i\beta\frac{z-R}{z+R}\right) + \Phi_1(z)$$
$$\Omega(z) = -\frac{2ih\cos(\varphi/2)\,z}{(z+R)(1+\varkappa)X_0(z)}\sum_{m=0}^{\infty}\left[-\varkappa N_m(z) + \frac{\mu}{\mu_0}R_m(z)\right] \times$$
$$T_m\left(i\beta\frac{z-R}{z+R}\right) + \Omega_1(z)$$
$$N_m(z) = x_m + \bar{x}_m - \frac{R^2}{z^2}y_m,$$
$$R_m(z) = \varkappa_0 x_m - \bar{x}_m + \frac{R^2}{z^2}y_m, \quad X_0(z) = \sqrt{(z-a)(z-b)}$$

($T_m(z)$ are Chebyshev polynomials of the first kind, $\Phi_1(z)$ and $\Omega_1(z)$ are bounded functions in the neighborhood of the ends of the inclusion).

Substituting (2.1) into (1.10) and (1.11), after some manipulation we arrive at the following infinite system of linear algebraic equations to determine the coefficients of the expansions x_m and y_m

$$(a_1x_n + a_2\bar{x}_n + a_3y_n)\frac{\pi}{2} + \sum_{m=0}^{\infty}(A^1_{n,\,n-1}x_m + {}^*A^1_{m,\,n-1}\bar{x}_m + B^1_{m,\,n-1}y_m + {}^*B^1_{m,\,n-1}\bar{y}_m) = P^1_{n-1} \quad (2.3)$$

$$(b_1x_n + b_2\bar{x}_n + b_3y_n)\frac{\pi}{2} + \sum_{m=0}^{\infty}(A^2_{m,\,n-1}x_m + {}^*A^2_{m,\,n-1}\bar{x}_m + B^2_{m,\,n-1}y_m + {}^*B^2_{m,\,n-1}\bar{y}_m) + Q_{n-1}\mu\varepsilon = P^2_{n-1}$$

$$\beta\sum_{m=1}^{\infty}x_m\{i[H(1,m-1)+H(1,m+1)] - 2\beta H(1,m)\} + x_0[\pi - 2\beta^2H(1,0)] = 0 \quad (2.4)$$

$$\beta\sum_{m=1}^{\infty}\{i(\bar{x}_m+y_m)[H(1,m-1)+H(1,m+1)] - 2\beta(\bar{x}_m - y_m)H(1,m)\} + (\bar{x}_0 - y_0)[\pi - 2\beta^2H(1,0)] = 0$$

Here

$$\begin{Vmatrix} A^1_{m,\,n} \\ {}^*A^1_{m,\,n} \\ A^2_{m,\,n} \\ {}^*A^2_{m,\,n}\end{Vmatrix} = 2\beta\begin{Vmatrix}1\\1\\ \varkappa_0\mu'\\ -\mu'\end{Vmatrix}\sum_{k=0}^{\infty}P_{2k}R(m-1,n,2k) \quad (m=1,2,\ldots)$$

$$\begin{vmatrix}B^1_{m,\,n}\\ B^2_{m,\,n}\end{vmatrix} = 2\beta\begin{vmatrix}1\\ -\mu'\end{vmatrix}T_{m,\,n} + \begin{vmatrix}a_3\\ b_3\end{vmatrix}S_{m,\,n}$$

$${}^*B^1_{m,\,n} = -{}^*B^2_{m,\,n} = \frac{2i\beta^2h(1-\mu')}{R(1+\varkappa)}P_n\{i\{H(1,m+1)+H(1,m-1) - 2\beta^2[H(2,m+1)+H(2,m-1)]\} + 4\beta[H(2,m)\beta^2 - H(1,m)]\}$$

$$\begin{Vmatrix} A^1_{0,\,n} \\ {}^*A^1_{0,\,n} \\ A^2_{0,\,n} \\ {}^*A^2_{0,\,n}\end{Vmatrix} = -2\beta\begin{Vmatrix}1\\1\\ \varkappa_0\mu'\\ -\mu'\end{Vmatrix}C_n + \begin{Vmatrix}a_1\\a_2\\b_1\\b_2\end{Vmatrix}G_n + \begin{Vmatrix}F_n^1\\F_n^2\\-F_n^1\\-F_n^2\end{Vmatrix}$$

$$\begin{vmatrix}B^1_{0,\,n}\\ B^2_{0,\,n}\end{vmatrix} = -2\beta\begin{vmatrix}1\\ -\mu'\end{vmatrix}V_n + \begin{vmatrix}a_3\\ b_3\end{vmatrix}W_n$$

$${}^*B^1_{0,\,n} = -{}^*B^2_{0,\,n} = \frac{i\beta h(1-\mu')}{R(1+\varkappa)}P_n\{\pi + 8\beta^2[H(2,0)\beta^2 - H(1,0)]\}$$

$$S_{m,n} = 2i\beta\{2\beta^2[H(2,\ m-n) + H(2,\ m-n-2) - H(2,\ m+n) - H(2,\ m+n+2)] - H(1,\ m-n) - H(1,\ m-n-2) + H(1,\ m+n) + H(1,\ m+n+2)\} + 4\beta^2\{\beta^2[H(2,m-n-1) - H(2,m+n+1)] - H(1,m-n-1) + H(1,m+n+1)\} - i\beta^2[P_n - i\beta(L_{n+1}+L_{n-1}) - 2\beta^2L_n]\{i[H(1,\ m+1) + H(1,\ m-1)] + 2\beta H(1,\ m)\}$$

$$T_{m,\,n} = \sum_{k=0}^{\infty}[P_{2k} - 8\beta^2(L_{2k} - \beta^2Z_{2k})]R(m-1,n,2k) - 2i\beta\sum_{k=1}^{\infty}\{L_{2k} + L_{2(k-1)} - 2\beta^2[Z_{2k} + Z_{2(k-1)}]\}R(m-1,n,2k-1)$$

$$V_n = C_n + 4\beta^2 \sum_{k=1}^{\infty} E_{2k-1} \{[H(3,2k-n-1) - H(3,2k+n+1)]\beta^2 -$$
$$H(2,2k-n-1) + H(2,2k+n+1)\} -$$
$$2i\beta \sum_{k=1}^{\infty} E_{2k-1} \{H(1,2k-n) + H(1,2k-n-2) - H(1,2k+n) -$$
$$H(1,2k+n+2) - 2\beta^2 [H(2,2k-n) + H(2,2k-n-2) -$$
$$H(2,2k+n) - H(2.2k+n+2)]\}$$

$$W_n = \frac{\pi}{2} [D_n - 3i\beta P_n - 2\beta^2 (L_{n+1} + L_{n-1}) + 4i\beta^3 L_n] +$$
$$i\beta [P_n - i\beta (L_{n+1} + L_{n-1}) - 2\beta^2 L_n] \{\pi + 8\beta^2 [\beta^2 H(2,0) - H(1,0)]\}$$

$$C_n = \frac{1}{2} \sum_{k=1}^{\infty} E_{2k-1} [H(1,2k-n-1) - H(1,2k+n+1)]$$

$$R(m,n,k) = -4(n+1)(k+1)[(m+1)^2 -$$
$$(n-k)^2]^{-1} [(m+1)^2 - (n+k+2)^2]^{-1} \cos^2 \left(\frac{m+n+k}{2}\pi\right)$$

$$G_n = \frac{\pi}{2}(D_n - i\beta P_n), \quad Q_n = -2\beta\pi P_n i,$$

$$F_n{}^1 = -\frac{\pi i\beta h(1-\mu')}{R(1+\varkappa)} P_n, \quad F_n{}^2 = -\frac{\pi i\beta h(1+\varkappa_0\mu')}{R(1+\varkappa)} P_n$$

$$P_{2k} = (1-p^2)(-p)^k \beta^{-2}, \quad P_{2k+1} = 0,$$

$$D_{2k+1} = \frac{1}{2}(1-p^2)^2(-p)^k \beta^{-2}, \quad D_{2k} = 0$$

$$L_{2k} = \frac{16(-1)^k p^{k+2}}{(1-p)^3(1+p)} [1 + k(1-p)], \quad L_{2k+1} = 0$$

$$Z_{2k} = 32(-1)^k p^{k+3}(1+p)^2(1-p^2)^{-5} \{2[(1+p)^2 - p] +$$
$$k(1-p)[k(1-p^2) + 2 + (1+p)^2]\},$$

$$Z_{2k+1} = 0 \quad \left(p = \operatorname{tg}^2 \frac{\varphi}{4}, k = 0, 1, 2, \ldots\right)$$

$$H(n,m) = [1 + (-1)^m] \pi \left(\frac{q^2-1}{2}\right)^{1-2n} q^{n-|m|/2} (-1)^n \times$$
$$\sum_{j=0}^{n-1} \binom{n+|m|/2-1}{j} \binom{2n-j-2}{n-1} (q^2-1)^j \quad \left(q = -\operatorname{ctg}^2 \frac{\varphi}{4}\right)$$

$$E_{2k-1} = \frac{16k}{\pi(4k^2-1)^2}, \quad E_{2(k-1)} = 0 \quad (k = 1, 2, 3, \ldots)$$

$$P_n{}^1 = -2\beta \int_{-1}^{1} \left[2\Gamma - \bar{\Gamma}' \left(\frac{-x+i\beta}{x+i\beta}\right)^2 - P_0{}^1(x)\right] \frac{\sqrt{1-x^2}\, U_n(x)}{x^2+\beta^2} dx$$

$$P_0{}^1(x) = K_0 + \bar{K}_0 - M_0 \left(\frac{-x+i\beta}{x+i\beta}\right)^2$$

$$P_n{}^2 = -2\beta \int_{-1}^{1} \left[(\varkappa - 1)\Gamma + \bar{\Gamma}' \left(\frac{-x+i\beta}{x+i\beta}\right)^2 -\right.$$
$$\left. P_0{}^2(x)\right] \frac{\sqrt{1-x^2}\, U_n(x)}{x^2+\beta^2} dx$$

$$P_0{}^2(x) = \left[\varkappa_0 K_0 - \bar{K}_0 + M_0 \left(\frac{-x+i\beta}{x+i\beta}\right)^2\right] \mu', \quad \mu' = \frac{\mu}{\mu_0}$$

The quasi-regualarity of the infinite system of linear algebraic equations (2.3) and (2.4) was investigated numerically for different parameters of the problem.

Proceeding in the same manner as in /4/, we take values for the constants K_0 and M_0 such that possible particular cases would follow from the solution of the problem

$$P_0{}^1(x) = \left[2\Gamma - \bar{\Gamma}' \left(\frac{-x+i\beta}{x+i\beta}\right)^2\right] \frac{\min(\mu, \mu_0)}{\mu}$$
$$P_0{}^2(x) = \left[(\varkappa - 1)\Gamma + \bar{\Gamma}' \left(\frac{-x+i\beta}{x+i\beta}\right)^2\right] \frac{\min(\mu, \mu_0)}{\mu_0}$$

Then

$$P_n^1 = -\pi\beta\left[1 - \frac{\min(\mu, \mu_0)}{\mu}\right]\{2\Gamma P_n - \bar{\Gamma}'[P_n - 2i\beta(L_{n+1} + L_{n-1}) -$$
$$8\beta^2 L_n + 4i\beta^3(Z_{n+1} + Z_{n-1}) + 8\beta^4 Z_n]\}$$
$$P_n^2 = -\pi\beta\left[1 - \frac{\min(\mu, \mu_0)}{\mu_0}\right]\{(\varkappa - 1)\Gamma P_n +$$
$$\bar{\Gamma}'[P_n - 2i\beta(L_{n+1} + L_{n-1}) - 8\beta^2 L_n + 4i\beta^3(Z_{n+1} + Z_{n-1}) + 8\beta^4 Z_n]\}$$

The angle of turning of the inclusion ε is found from the condition (Λ is the domain of the inclusion)

$$\operatorname{Re}\int_{\Lambda}\frac{R^2}{z}\left[\Phi(z) - \bar{\Omega}\left(\frac{R^2}{z}\right)\right]dz = 0$$

which becomes after manipulation

$$2\operatorname{Re}x_0 - \operatorname{Re}\sum_{m=1}^{\infty} y_m\{-2i\beta\{H(1, m-1) + H(1, m+1) - \tag{2.5}$$
$$2\beta^2[H(2, m-1) + H(2, m+1)]\} + 8\beta^2[\beta^2 H(2, m) - H(1, m)]\} -$$
$$\operatorname{Re}y_0\{\pi + 8\beta^2[\beta^2 H(2,0) - H(1,0)]\} = 0$$

Following /5/, the state of stress of the plate in the neighborhood of the end of the inclusion can be represented in the polar coordinate system (ρ, γ) (Fig.1) in the following form:

$$\begin{Vmatrix}\sigma_\rho\\ \sigma_\gamma\\ \tau_{\rho\gamma}\end{Vmatrix} = K_1^*\begin{Vmatrix}5c_1 - c_3\\ 3c_1 - c_3\\ s_1 + s_3\end{Vmatrix} + K_2^*\begin{Vmatrix}-5s_1 + 3s_3\\ -3s_1 - 3s_3\\ c_1 + 3c_3\end{Vmatrix} +$$

$$K_3^*\begin{Vmatrix}5c_1 + (1 + 2\varkappa)c_3\\ 3c_1 - (1 + 2\varkappa)c_3\\ s_1 - (1 + 2\varkappa)s_3\end{Vmatrix} + K_4^*\begin{Vmatrix}-5s_1 + (1 - 2\varkappa)s_3\\ -3s_1 - (1 - 2\varkappa)s_3\\ c_1 + (1 - 2\varkappa)c_3\end{Vmatrix} + O(\rho^0)$$

$$K_i^* = \frac{K_i}{4\sqrt{2\rho}}, \quad i = 1, 2, 3, 4 \quad (s_1 = \sin{}^1/_2\gamma,$$
$$s_3 = \sin{}^3/_2\gamma, \quad c_1 = \cos{}^1/_2\gamma, \quad c_3 = \cos{}^3/_2\gamma)$$

Here K_i are stress intensity coefficients determined from the formulas

$$K_1^j - iK_2^j = -\frac{2h}{(1 + \varkappa)\sqrt{R\sin\varphi}}\frac{\mu}{\mu_0}\sum_{m=0}^{\infty} R_m(d_j)(-1)^{(m+1)(2-j)}$$

$$K_3^j - iK_4^j = -\frac{2h}{(1 + \varkappa)\sqrt{R\sin\varphi}}\sum_{m=0}^{\infty} N_m(d_j)(-1)^{(m+1)(2-j)},$$

$$d_j = \begin{cases}a, & j = 1\\ b, & j = 2\end{cases}$$

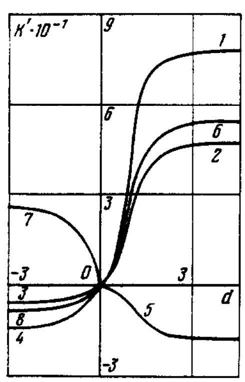

Fig.2

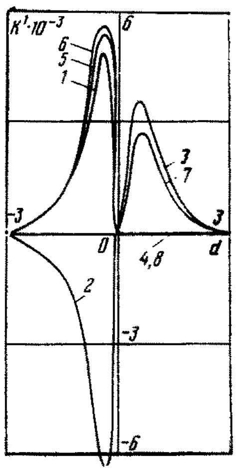

Fig.3

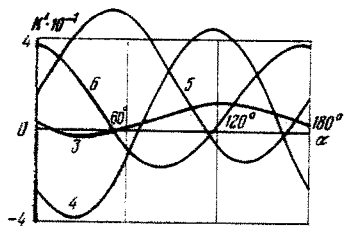

Fig.4

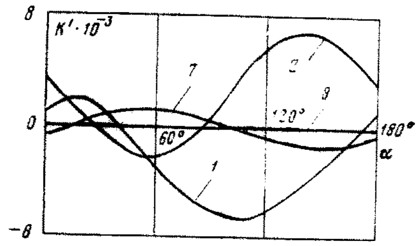

Fig.5

($j = 1$ for the end a and $j = 2$ for the end b).

3. Let us examine certain limit cases.

Plate with a crack along the arc of a circle ($\mu_0 \to 0$). Let us make the following replacement in (1.8)

$$\Phi_0(t) = \mu_0/\mu\, \Phi_0^*(t),\ \Gamma_0(t) = \mu_0/\mu \Gamma_0^*(t)$$

We allow $\mu_0 \to 0$, and substitute the solution of the singular integral equation obtained into (1.7), to find expressions for the functions $\Phi(z)$ and $\Omega(z)$ that agree with those presented in /2/.

Homogeneous plate ($\mu = \mu_0,\ \varkappa = \varkappa_0$). The case of a homogeneous plate can be obtained by two methods: by passing to the limit in (1.7) as $h \to 0$, or by passing to the limit in (2.3)–(2.5) as $\mu \to \mu_0$ (in this case the homogeneous system of linear algebraic equations yields the solution $\varepsilon = x_m = y_m = 0$ $(m = 0, 1, \ldots)$, i.e., $A(t) = B(t) = 0$). In both cases we have

$$\Phi(z) = \Gamma, \quad \Omega(z) = \Gamma - \bar{\Gamma}' R^2/z^2$$

Plate with a thin-walled absolutely rigid inclusion along the arc of a circle ($\mu_0 \to \infty$). Passing to the limit in (1.8) as $\mu_0 \to \infty$, solving the singular integral equation obtained, and substituting this solution into (1.7), we find expressions for the functions $\Phi(z)$ and $\Omega(z)$ that agree with those presented in /6/.

Plate with a rectilinear thin-walled elastic inclusion ($\varphi \to 0$, but $R\varphi \to l = \mathrm{const}$). Letting $\varphi \to 0\ (R\varphi \to l)$ in (1.8) and introducing the notation

$$M(t) = \varkappa_0 \Phi_0(t) - \overline{\Phi_0(t)} + \Gamma_0(t),\quad K(t) = \Phi_0(t) + \overline{\Phi_0(t)} - \Gamma_0(t)$$

we arrive at a system of two singular integro-differential equations of Prandtl type for the rectilinear thin-walled elastic inclusion /4/.

4. The solution of the problem was analyzed numerically on the ES-1022 electronic computer, and the results are represented in Figs.2–5. The calculations were performed for the following values of the parameters

$$\sigma_1 = P,\ \sigma_2 = 0,\ h/(R \sin\varphi) = 0{,}1,\ \nu = \nu_0 = 1/3.$$

The dependence of the stress intensity coefficients $K_i' = K_i/(P\sqrt{R \sin\varphi})$ at the point b on the relative stiffness $d = \lg \mu/\mu_0$ of the plate and inclusion for $\alpha = 0$ is shown on Figs.2 and 3. Values of K_i' for an inclusion aperture of $\varphi = \pi/6$ correspond to the curves of i $(i = 1, 2, 3, 4)$, and to the curves $i + 4$ for the aperture angle $\varphi = \pi/2$. In the limit cases of the problem under investigation, the numerical values agree with the results obtained on the basis of /2, 6,7/.

The dependence of K_i' at the same point on the angle α for an inclusion aperture angle of $\varphi = \pi/3$ is represented in Figs.4 and 5. The value of the relative stiffness $d = -1$ corresponds to the curves of i $(i = 1, 2, 3, 4)$ and of $d = 1$ to the curve $i + 4$. Let us note that the curves in Figs.3 and 5 are continuations of the corresponding curves in Figs.2 and 4, but only in another scale.

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